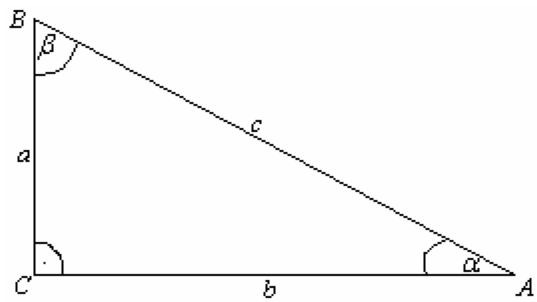
TRIGONOMETRIJSKE FUNKCIJE OŠTROG UGLA

Trigonometrija je prvobitno predstavljala oblast matematike koje se bavila izračunavanjem nepoznatih elemenata trougla pomoću poznatih. Sam njen naziv potiče od dve grčke reči TRIGONOS- što znači trougao i METRON- što znači mera. Kako se definišu trigonometrijske funkcije?

Posmatrajmo pravougli trougao ABC.



$a,b \rightarrow$ katete
$c \rightarrow$ hipotenuza
$a^2 + b^2 = c^2 \rightarrow$ Pitagorina teorema

$$\sin\alpha = \frac{naspramna\ \ kateta}{hipotenuza} = \frac{a}{c}$$

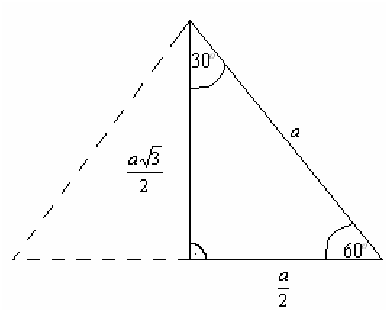
$$\cos\alpha = \frac{nalegla\ \ kateta}{hipotenuza} = \frac{b}{c}$$

$$tg\alpha = \frac{naspramna\ \ kateta}{nalegla\ \ kateta} = \frac{a}{b}$$

$$ctg\alpha = \frac{nalegla\ \ kateta}{naspramna\ \ kateta} = \frac{b}{a}$$

PAZI: Sam simbol sin,cos,tg,ctg sam za sebe ne označava nikakvu veličinu! Uvek mora da ima i ugao.

Izračunajmo vrednost trigonometrijskih funkcija za uglove od $30^o, 45^o$ ***i*** 60^o. Najpre ćemo posmatrati polovinu jednakostraničnog trougla.



Kao što znamo visina jednakostraničnog trougla je :

$$h = \frac{a\sqrt{3}}{2}$$

$$\sin 30^o = \frac{naspramna\ kateta}{hipotenuza} = \frac{\frac{a}{2}}{a} = \frac{a}{2a} = \frac{1}{2}$$

$$\cos 30^o = \frac{nalegla\ kateta}{hipotenuza} = \frac{\frac{a\sqrt{3}}{2}}{a} = \frac{\sqrt{3}}{2}$$

$$tg30^o = \frac{naspramna\ kateta}{nalegla\ kateta} = \frac{\frac{a}{2}}{\frac{a\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}(racionališemo) = \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$ctg30^o = \frac{nalegla\ kateta}{naspramna\ kateta} = \frac{\frac{a\sqrt{3}}{2}}{\frac{a}{2}} = \sqrt{3}$$

Sada ćemo uraditi (po definiciji) i za ugao od 60^o.

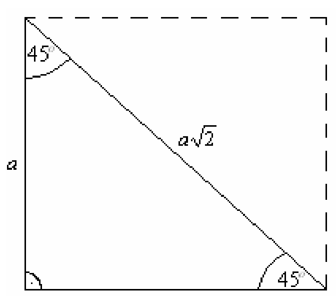
$$\sin 60^o = \frac{\frac{a\sqrt{3}}{2}}{a} = \frac{\sqrt{3}}{2}$$

$$\cos 60^o = \frac{\frac{a}{2}}{a} = \frac{1}{2}$$

$$tg60^o = \frac{\frac{a\sqrt{3}}{2}}{\frac{a}{2}} = \sqrt{3}$$

$$ctg60^o = \frac{\frac{a}{2}}{\frac{a\sqrt{3}}{2}} = \frac{\sqrt{3}}{3}$$

Za vrednost trigonometrijskih funkcija ugla od 45^o upotrebićemo polovinu kvadrata.



Kao što znamo dijagonala kvadrata je $d = a\sqrt{2}$

$$\sin 45^o = \frac{naspramna\ kateta}{hipotenuza} = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\cos 45^o = \frac{nalegla\ kateta}{hipotenuza} = \frac{a}{a\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$tg 45^o = \frac{naspramna\ kateta}{nalegla\ kateta} = \frac{a}{a} = 1$$

$$ctg 45^o = \frac{nalegla\ kateta}{naspramna\ kateta} = \frac{a}{a} = 1$$

Na ovaj način smo dobili tablicu:

	$\alpha = 30^o$	$\alpha = 45^o$	$\alpha = 60^o$
sinα	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
cosα	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
tgα	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$
ctgα	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$

Naravno, kasnije ćemo tablicu proširiti na sve uglove od $0^o \to 360^o$.

Osnovni trigonometrijski indetiteti:

1) $\sin^2 \alpha + \cos^2 \alpha = 1$
2) $tg\alpha = \frac{\sin \alpha}{\cos \alpha}$
3) $ctg\alpha = \frac{\cos \alpha}{\sin \alpha}$
4) $tg\alpha \cdot ctg\alpha = 1$

Da probamo da dokažemo neke od indetiteta:

1) $\sin^2 \alpha + \cos^2 \alpha =$(pogledajmo definicije: $\sin \alpha = \frac{a}{c}$ i $\cos \alpha = \frac{b}{c}$; to da zapamtimo)=
$\frac{a^2}{c^2} + \frac{b^2}{c^2} = \frac{a^2 + b^2}{c^2}$=(važi Pitagorina teorema, $a^2 + b^2 = c^2$) $= \frac{c^2}{c^2} = 1$

2) $\dfrac{\sin\alpha}{\cos\alpha}=\dfrac{\frac{a}{c}}{\frac{b}{c}}=\dfrac{a\cdot c}{b\cdot c}=\dfrac{a}{b}=tg\alpha$ slično se dokazuje i za $ctg\alpha$

4) $tg\alpha\cdot ctg\alpha=$(zamenimo iz definicije, da je $tg\alpha=\dfrac{a}{b}$ i $ctg\alpha=\dfrac{b}{a}$)$=\dfrac{a}{b}\cdot\dfrac{b}{a}=1$

Baš lako, zar ne?

Iz osnovnih indetiteta se mogu izvesti razne druge jednakosti:

1) Ako krenemo od:

$\sin^2\alpha+\cos^2\alpha=1\rightarrow$ ovo delimo sa $\cos^2\alpha$

$$\frac{\sin^2\alpha}{\cos^2\alpha}+\frac{\cos^2\alpha}{\cos^2\alpha}=\frac{1}{\cos^2\alpha}$$

$tg^2\alpha+1=\dfrac{1}{\cos^2\alpha}\rightarrow$ Odavde izrazimo $\cos^2\alpha$

$$\boxed{\cos^2\alpha=\frac{1}{tg^2\alpha+1}}$$

Ako sad ovo zamenimo u:

$$\sin^2\alpha+\cos^2\alpha=1$$

$$\sin^2\alpha+\frac{1}{tg^2\alpha+1}=1$$

$$\sin^2\alpha=1-\frac{1}{tg^2\alpha+1}$$

$$\sin^2\alpha=\frac{tg^2\alpha+1-1}{tg^2\alpha+1}$$

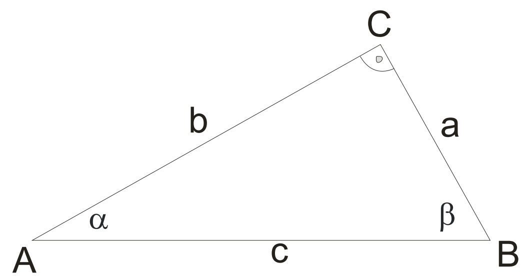
$$\boxed{\sin^2\alpha=\frac{tg^2\alpha}{tg^2\alpha+1}}$$

Ove dve identičnosti ćemo zapisati i koristiti ih u zadacima!!!

Još jedna stvar, da izvedemo i trigonometrijske funkcije komplementnog ugla. Kako je kod pravouglog trougla $\alpha+\beta=90^o$ tj. komplementni su, važi:

$$\sin(90^o - \alpha) = \cos\alpha \qquad \text{tj.} \qquad \sin\beta = \cos\alpha$$
$$\cos(90^o - \alpha) = \sin\alpha \qquad \cos\beta = \sin\alpha$$
$$tg(90^o - \alpha) = ctg\alpha \qquad tg\beta = ctg\alpha$$
$$ctg(90^o - \alpha) = tg\alpha \qquad ctg\beta = tg\alpha$$

Odakle ovo?



sa slike (po definiciji) je

$$\sin\alpha = \frac{a}{c} \qquad \sin\beta = \frac{b}{c}$$
$$\cos\alpha = \frac{b}{c} \qquad \cos\beta = \frac{a}{c}$$
$$tg\alpha = \frac{a}{b} \qquad tg\beta = \frac{b}{a}$$
$$ctg\alpha = \frac{b}{a} \qquad ctg\beta = \frac{a}{b}$$

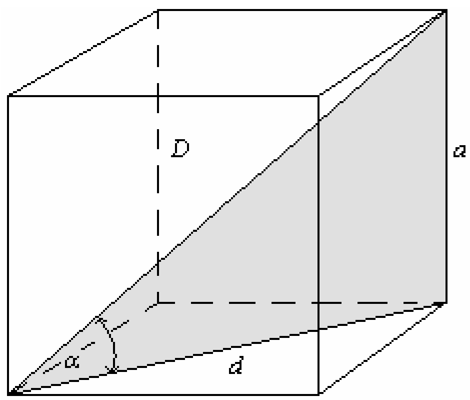
Primeri:

1) Date su katete pravouglog trougla *a*=8cm i *b*=6cm. Odrediti vrednost svih trigonometrijskih funkcija uglova α i β

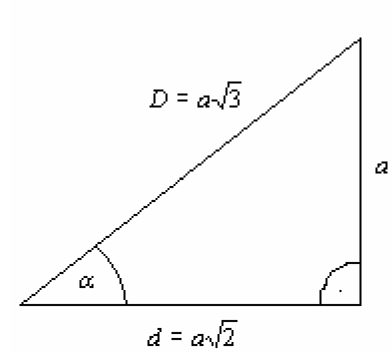
$$a = 8cm$$
$$b = 6cm$$
$$c^2 = a^2 + b^2$$
$$c^2 = 8^2 + 6^2$$
$$c^2 = 64 + 36$$
$$c^2 = 100$$
$$c = 10cm$$

$$\sin\alpha = \frac{a}{c} = \frac{8}{10} = \frac{4}{5} = \cos\beta$$
$$\cos\alpha = \frac{b}{c} = \frac{6}{10} = \frac{3}{5} = \sin\beta$$
$$tg\alpha = \frac{a}{b} = \frac{8}{6} = \frac{4}{3} = ctg\beta$$
$$ctg\alpha = \frac{b}{a} = \frac{6}{8} = \frac{3}{4} = tg\beta$$

2) Izračunati vrednost trigonometrijskih funkcija nagibnog ugla dijagonale kocke prema osnovi.



Izvučemo na stranu ovaj trougao:



Kao što znamo mala dijagonala je $d = a\sqrt{2}$, a velika dijagonala (telesna) $D = a\sqrt{3}$. Po definicijama je:

$$\sin\alpha = \frac{a}{a\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\cos\alpha = \frac{a\sqrt{2}}{a\sqrt{3}} = \frac{\sqrt{2}}{\sqrt{3}} = \frac{\sqrt{2}}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{6}}{3}$$

$$tg\alpha = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$ctg\alpha = \frac{a\sqrt{2}}{a} = \sqrt{2}$$

3) U pravouglom trouglu je $c = 24cm$ **i** $\sin\alpha = 0,8$ **. Odrediti katete.**

$c = 24cm$

$\sin\alpha = 0,8$

$a = ?$

$b = ?$

Po definiciji je:

$$\sin\alpha = \frac{a}{c}$$

$$0,8 = \frac{a}{24}$$

$$a = 24\cdot 0,8$$

$$a = 19,2cm$$

$b^2 = c^2 - a^2$ sad ide Pitagorina teorema

$$b^2 = 24^2 - (19,2)^2$$

$$b^2 = 576 - 368,64$$

$$b^2 = 207,36$$

$$b = 14,4cm$$

4) Izračunati vrednost ostalih trigonometrijskih funkcija ako je:

a) $\sin\alpha = 0{,}6$

b) $\cos\alpha = \frac{12}{13}$

v) $tg\alpha = 0{,}225$

Rešenje:

a) $\sin\alpha = \frac{3}{5}$ jer $0{,}6 = \frac{6}{10} = \frac{3}{5}$ .Najpre ćemo iskoristiti da je $\sin^2\alpha + \cos^2\alpha = 1$

$$\left(\frac{3}{5}\right)^2 + \cos^2\alpha = 1$$

$$\cos^2\alpha = 1 - \frac{9}{25}$$

$$\cos^2\alpha = \frac{16}{25}$$

$$\cos\alpha = \pm\sqrt{\frac{16}{25}}$$

$$\cos\alpha = \pm\frac{4}{5}$$

Pošto su oštri uglovi u pitanju:

$$\boxed{\cos\alpha = +\frac{4}{5}}$$

$$tg\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{\frac{3}{5}}{\frac{4}{5}} = \frac{3}{4}$$

$$ctg\alpha = \frac{1}{tg\alpha} = \frac{4}{3}$$

b)

$$\cos\alpha = \frac{12}{13}$$

$$\sin^2\alpha + \cos^2\alpha = 1$$

$$\sin^2\alpha + \left(\frac{12}{13}\right)^2 = 1$$

$$\sin^2\alpha = 1 - \frac{144}{169}$$

$$\sin^2\alpha = \frac{25}{169}$$

$$\sin\alpha = \pm\sqrt{\frac{25}{169}}$$

$$\sin\alpha = \pm\frac{5}{13}$$

oštar ugao, pa uzimamo +

$$\boxed{\sin\alpha = \frac{5}{13}}$$

$$tg\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{\frac{5}{13}}{\frac{12}{13}} = \frac{5}{12}$$

$$ctg\alpha = \frac{12}{5}$$

v) $tg\alpha = 0{,}225 = \frac{225}{1000} = \frac{9}{40}$

Iskoristićemo jednakosti: $\boxed{\sin^2\alpha = \frac{tg^2\alpha}{tg^2\alpha+1}}$ i $\boxed{\cos^2\alpha = \frac{1}{tg^2\alpha+1}}$

$$\sin^2\alpha = \frac{tg^2\alpha}{tg^2\alpha+1}$$

$$\sin^2\alpha = \frac{\left(\frac{9}{40}\right)^2}{\left(\frac{9}{40}\right)^2+1}$$

$$\sin^2\alpha = \frac{\frac{81}{1600}}{\frac{81}{1600}+1}$$

$$\sin^2\alpha = \frac{\frac{81}{1600}}{\frac{81+1600}{1600}}$$

$$\sin^2\alpha = \frac{81}{1681}$$

$$\sin\alpha = \pm\sqrt{\frac{81}{1681}}$$

$$\sin\alpha = \pm\frac{9}{41}$$

$$\sin\alpha = +\frac{9}{41}$$

$$\cos^2\alpha = \frac{1}{tg^2\alpha+1}$$

$$\cos^2\alpha = \frac{1}{\frac{1681}{1600}} \rightarrow \cos^2\alpha = \frac{1600}{1681}$$

$$\cos\alpha = \pm\sqrt{\frac{1600}{1681}} \rightarrow \cos\alpha = \pm\frac{40}{41}$$

$$\cos\alpha = +\frac{40}{41}$$

Za kotangens je lako:

$$ctg\alpha = \frac{1}{tg\alpha}$$

$$ctg\alpha = \frac{40}{9}$$

5) Izračunaj vrednosti ostalih trigonometrijskih funkcija ako je:

a) $\sin\alpha = \dfrac{a^2-9}{a^2+9}$

b) $ctg\alpha = \dfrac{a^2-4}{4a}$

a)

$$\sin^2\alpha + \cos^2\alpha = 1$$
$$\cos^2\alpha = 1 - \sin^2\alpha$$
$$\cos^2\alpha = 1 - \left(\frac{a^2-9}{a^2+9}\right)^2$$
$$\cos^2\alpha = 1 - \frac{(a^2-9)^2}{(a^2+9)^2}$$
$$\cos^2\alpha = \frac{(a^2+9)^2-(a^2-9)^2}{(a^2+9)^2}$$
$$\cos^2\alpha = \frac{a^4+18a^2+81-a^4+18a^2-81}{(a^2+9)^2}$$

$$\cos^2\alpha = \frac{36a^2}{(a^2+9)^2}$$
$$\cos\alpha = \sqrt{\frac{36a^2}{(a^2+9)^2}}$$
$$\cos\alpha = \frac{6a}{a^2+9}$$

$$tg\alpha = \frac{\sin\alpha}{\cos\alpha}$$
$$tg\alpha = \frac{\frac{a^2-9}{\cancel{a^2+9}}}{\frac{6a}{\cancel{a^2+9}}}$$
$$tg\alpha = \frac{a^2-9}{6a}$$
$$ctg\alpha = \frac{6a}{a^2-9}$$

b)

$$ctg\alpha = \frac{a^2-4}{4a} \Rightarrow tg\alpha = \frac{4a}{a^2-4}$$
$$\sin^2\alpha = \frac{tg^2\alpha}{tg^2\alpha+1}$$
$$\sin^2\alpha = \frac{\left(\frac{4a}{a^2-4}\right)^2}{\left(\frac{4a}{a^2-4}\right)^2+1}$$
$$\sin^2\alpha = \frac{\frac{16a^2}{(a^2-4)^2}}{\frac{16a^2}{(a^2-4)^2}+1}$$
$$\sin^2\alpha = \frac{16a^2}{16a^2+a^4-8a^2+16}$$
$$\sin^2\alpha = \frac{16a^2}{a^4+8a^2+16}$$
$$\sin\alpha = \sqrt{\frac{16a^2}{(a^2+4)^2}}$$
$$\sin\alpha = \frac{4a}{a^2+4}$$

$$\cos^2\alpha = \frac{1}{tg^2\alpha+1}$$
$$\cos^2\alpha = \frac{1}{\left(\frac{4a}{a^2-4}\right)^2+1}$$
$$\cos^2\alpha = \frac{1}{\frac{16a^2+(a^2-4)^2}{(a^2-4)^2}}$$
$$\cos^2\alpha = \frac{1}{\frac{(a^2+4)^2}{(a^2-4)^2}}$$
$$\cos^2\alpha = \frac{(a^2-4)^2}{(a^2+4)^2}$$
$$\cos\alpha = \sqrt{\frac{(a^2-4)^2}{(a^2+4)^2}}$$
$$\cos\alpha = \frac{a^2-4}{a^2+4}$$

6) **Dokazati identitet** $\left(1+tgx+\frac{1}{\cos x}\right)\cdot\left(1+tgx-\frac{1}{\cos x}\right)=2tgx$

$$\left(1+tgx+\frac{1}{\cos x}\right)\cdot\left(1+tgx-\frac{1}{\cos x}\right)=$$
$$\left(1+\frac{\sin x}{\cos x}+\frac{1}{\cos x}\right)\cdot\left(1+\frac{\sin x}{\cos x}-\frac{1}{\cos x}\right)=$$
$$\frac{\cos x+\sin x+1}{\cos x}\cdot\frac{\cos x+\sin x-1}{\cos x}=$$ gore je razlika kvadrata
$$\frac{(\cos x+\sin x)^2-1^2}{\cos^2 x}=$$(jedinicu ćemo zameniti sa $\sin^2 x+\cos^2 x$)
$$\frac{\cos^2 x+2\cos x\sin x+\sin^2 x-\sin^2 x-\cos^2 x}{\cos^2 x}=\frac{2\cancel{\cos x}\sin x}{\cos^{\cancel{2}} x}=$$
$$=2\frac{\sin x}{\cos x}=2tgx$$

7) Dokazati da je:

a) $\cos^2 18^o+\cos^2 36^o+\cos^2 54^o+\cos^2 72^o=2$

Pošto važi da kad je $\alpha+\beta=90^o$ $\cos\alpha=\sin\beta$, $\cos 54^o$ ćemo zameniti sa $\sin 36^o$ a $\cos 72^o$ ćemo zameniti sa $\sin 18^o$. Onda je:

$$\cos^2 18^o+\cos^2 36^o+\cos^2 54^o+\cos^2 72^o=$$
$$\cos^2 18^o+\cos^2 36^o+\sin^2 36^o+\sin^2 18^o=$$

$=1+1=2$

b)

$$tg1^o\cdot tg2^o\cdot tg3^o...tg44^o\cdot tg45^o\cdot tg46^o...tg89^o=1$$

Kako je $tg\alpha=ctg\beta$ za $(\alpha+\beta=90^o)$ biće:

$$tg1^o\cdot tg2^o\cdot tg3^o...tg44^o\cdot tg45^o\cdot ctg44^o...ctg2^o\cdot ctg1^o$$

= Kako je $tg\alpha\cdot ctg\alpha=1$

$=1\cdot 1\cdot ...\cdot tg45^o=1$

8) Dokazati identitet $\dfrac{3}{1-\sin^6\alpha-\cos^6\alpha}=(tg\alpha+ctg\alpha)^2$

$\dfrac{3}{1-\sin^6 x-\cos^6 x}=\dfrac{3}{1-(\sin^6 x+\cos^6 x)}=$ Pokušaćemo da transformišemo izraz

$\sin^6 x-\cos^6 x$ Podjimo od $\sin^2 x-\cos^2 x=1$ pa "dignemo'' na treći stepen:

$(A+B)^3=A^3+3A^2B+3AB^2+B^3$

$\sin^2 x+\cos^2 x=1............/()^3$

$\sin^6 x+3\sin^4 x\cos^2 x+3\sin^2 x\cos^4 x+\cos^6 x=1$

$\sin^6 x+3\sin^2 x\cos^2 x\underbrace{(\sin^2 x+\cos^2 x)}_{1}+\cos^6 x=1$

Dakle: $\sin^6 x+\cos^6 x=1-3\sin^2 x\cos^2 x$

Vratimo se u zadatak:

$$=\frac{3}{1-1+3\sin^2 x\cos^2 x}=\frac{3}{3\sin^2 x\cos^2 x}=\frac{1}{\sin^2 x\cos^2 x}$$

Da vidimo sad desnu stranu:

$$\begin{aligned}(tg\alpha+ctg\alpha)^2 &= tg^2\alpha+2tg\alpha ctg\alpha+ctg^2\alpha\\ &=\frac{\sin^2\alpha}{\cos^2\alpha}+2+\frac{\cos^2\alpha}{\sin^2\alpha}\\ &=\frac{\sin^4\alpha+2\sin^2\alpha\cos^2\alpha+\cos^4\alpha}{\sin^2\alpha\cos^2\alpha}\\ &=\frac{(\sin^2\alpha+\cos^2\alpha)^2}{\sin^2\alpha\cos^2\alpha}\\ &=\frac{1}{\sin^2\alpha\cos^2\alpha}\end{aligned}$$

Ovim smo dokazali da su leva i desna strana jednake:

Uslov je

$1-\sin^6\alpha-\cos^6\alpha\neq 0$

$\sin^6\alpha-\cos^6\alpha\neq 1$

$1-3\sin^2\alpha\cos^2\alpha\neq 1$

$\sin^2\alpha\cos^2\alpha\neq 0$

$\sin\alpha\neq 0\wedge\cos\alpha\neq 0$

9) Dokazati identitet: $(tg^3\alpha+\frac{1-tg\alpha}{ctg\alpha}):(\frac{1-ctg\alpha}{tg\alpha}+ctg^3\alpha)=tg^4\alpha$

Kao i obično, krenemo od teže strane dok ne dodjemo do lakše...

$$(tg^3\alpha+\frac{1-tg\alpha}{ctg\alpha}):(\frac{1-ctg\alpha}{tg\alpha}+ctg^3\alpha)=$$

$$(tg^3\alpha+\frac{1-tg\alpha}{\frac{1}{tg\alpha}}):(\frac{1-\frac{1}{tg\alpha}}{tg\alpha}+\frac{1}{tg^3\alpha})=$$

$$(tg^3\alpha+tg\alpha\cdot(1-tg\alpha)):(\frac{\frac{tg\alpha-1}{tg\alpha}}{tg\alpha}+\frac{1}{tg^3\alpha})=$$

$$(tg^3\alpha+tg\alpha-tg^2\alpha):(\frac{tg\alpha-1}{tg^2\alpha}+\frac{1}{tg^3\alpha})=$$

$$(tg^3\alpha-tg^2\alpha+tg\alpha):(\frac{tg\alpha(tg\alpha-1)+1}{tg^3\alpha})=$$

$$tg\alpha(tg^2\alpha-tg\alpha+1):(\frac{tg^2\alpha-tg\alpha+1}{tg^3\alpha})=\frac{tg\alpha(tg^2\alpha-tg\alpha+1)}{1}:(\frac{tg^2\alpha-tg\alpha+1}{tg^3\alpha})=$$

$$\frac{tg\alpha\,\cancel{(tg^2\alpha-tg\alpha+1)}}{1}\cdot\frac{tg^3\alpha}{\cancel{tg^2\alpha-tg\alpha+1}}=tg\alpha\cdot tg^3\alpha=\boxed{tg^4\alpha}$$

Naravno, uslovi zadatka su da (pošto u imeniocu ne sme da bude nula):

$tg\alpha\neq 0$ **i** $ctg\alpha\neq 0$